

# Wu-Yang ambiguity in connection space

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Two distinct gauge potentials can have the same field strength, in which case they are said to be “copies” of each other. The consequences of this possibility for the general space  $\mathcal{A}$  of gauge potentials are examined. Any two potentials are connected by a straight line in  $\mathcal{A}$ , but a straight line going through two copies either contains no other copy or is entirely formed by copies.

## I. INTRODUCTION

A good understanding of the Wu-Yang ambiguity [1], with all its aspects and consequences, does not seem to have been as yet achieved. Activity on the subject has been intensive in the first years after its discovery [2, 3, 4, 5, 6, 7] and declined afterwards [8]. Progress has been made step by step, sometimes through the discovery of general properties of formal character [9], most of times by unearthing particular cases which elucidate special points [10]. This note is devoted to the presentation of one more formal property, which shows up in the space  $\mathcal{A}$  of gauge potentials, or connections (“A-space”). We shall mostly use invariant notation, such as  $A = J_a A^a_\mu dx^\mu$  for gauge potentials (connections),  $F = \frac{1}{2} J_a F^a_{\mu\nu} dx^\mu \wedge dx^\nu$  for the field strength (curvatures),  $K = J_a K^a_\mu dx^\mu$  for covectors in the adjoint representation of the gauge algebra, etc. Wedge products will be implicit.

Consider two connection forms  $A$  and  $A^\sharp$ , with curvatures

$$F = dA + AA \quad \text{and} \quad F^\sharp = dA^\sharp + A^\sharp A^\sharp.$$

Under a gauge transform given by a member of the gauge group represented by  $g$  in the adjoint representation, the connections will transform according to

$$A \Rightarrow A' = gAg^{-1} + gdg^{-1} \quad ; \quad A^\sharp \Rightarrow A^{\sharp'} = gA^\sharp g^{-1} + gdg^{-1}.$$

The curvatures are 2-forms covariant under these transformations:  $F \Rightarrow F' = gFg^{-1}$ ,  $F^\sharp \Rightarrow F^{\sharp'} = gF^\sharp g^{-1}$ . The difference

$$K = A^\sharp - A$$

will be a 1-form transforming according to

$$K' = A^{\sharp'} - A' = g(A^\sharp - A)g^{-1} = g^{-1}Kg.$$

That is, the difference between two connections is a covariant 1-form in the adjoint representation. The ambiguity appears when  $F^\sharp = F$  but  $K \neq 0$ . Gauge potentials like  $A$  and  $A^\sharp$ , corresponding to the same field strength, are frequently called “copies”.

It is well known that gauge covariance divides  $\mathcal{A}$  into equivalence classes, each class representing a potential up to gauge transformations. The space of gauge inequivalent connections is  $\alpha = \mathcal{A}/\mathcal{G}$ , where  $\mathcal{G}$  is the so-called large group of point-dependent group elements [11, 12]. We shall not enter into the details of the  $\mathcal{A}$ -space structure [13]. Let us only say that, technically, only variations not along the large group are of interest for copies. This excludes gauge transformations.

The whole point is that  $F$  does not determine  $A$ . At each point of spacetime a gauge can be chosen in which  $A = 0$  and consequently  $F = dA$ . This is true also along a line. One might think of integrating by the homotopy formula [14] to obtain  $A$  from  $F$ . This is impossible because the involved homotopy requires the validity of  $F = dA$  on a domain of the same dimension of spacetime and the alluded gauge cannot exist (unless  $F = 0$ ) on a domain of dimension 2 or higher [15]. For copies, the difference form  $K$  defines a translation on space  $\mathcal{A}$  leaving  $F$  invariant. This invariance establishes another division of  $\mathcal{A}$  in equivalence classes. In effect, define the relation  $R$  by:  $AA^\sharp$  if  $A^\sharp$  is a copy of  $A$ . This relation is reflexive, transitive and symmetric, consequently an equivalence. The space of connections with distinct curvatures will be the quotient  $\alpha/R$ .

The gauge group element  $g$  can be seen as a matrix acting on column-vectors  $V$  belonging to an associated vector representation. The covariant differentials according to  $A$  and  $A^\sharp$  will have, in the vector representation, the forms  $DV = dV + AV$ ;  $D^\sharp V = dV + A^\sharp V = dV + (A^\sharp - A)V + AV$ , that is,

$$D^\sharp V = DV + KV.$$

For a matrix 1-form like the difference 1-form  $K$ ,

$$DK = dK + AK + KA = dK + \{A, K\};$$

$$D^\sharp K = dK + A^\sharp K + KA^\sharp = dK + \{A^\sharp, K\}.$$

It is immediately found that

$$D^\sharp K = DK + 2KK \quad (1)$$

and the relation between the two curvatures is

$$F^\sharp = F + DK + KK. \quad (2)$$

A direct calculation gives

$$DDK + [K, F] = 0, \quad (3)$$

which actually holds for any covariant 1-form in the adjoint representation.

Equation (2) leads to a general result: given a connection  $A$  defining a covariant derivative  $D_A$ , each solution  $K$  of  $D_A K + KK = 0$  will give a copy.

## II. ON THE CONNECTION SPACE $\mathcal{A}$

We have been making implicit use of one main property of the space  $\mathcal{A}$  of connections, namely:  $\mathcal{A}$  is a convex affine space, homotopically trivial [16]. One way to state this operationally [17] has been used above: given a connection  $A$ , every other connection  $A^\sharp$  can be written as  $A^\sharp = A + K$ , for some covariant covector  $K$ . Another way is: through any two connections  $A$  and  $A^\sharp$  there exists a straight line of connections  $A_t$ , given by

$$A_t = tA^\sharp + (1 - t)A. \quad (4)$$

In this expression  $t$  is a real parameter,  $A_0 = A$  and  $A_1 = A^\sharp$ . In terms of the difference form  $K$ , that straight line is written

$$A_t = A + tK = A^\sharp - (1 - t)K. \quad (5)$$

Of course,  $\frac{dA_t}{dt} = K$ . Indicating by  $D_t$  the covariant derivative according to connection  $A_t$ , we find

$$D_t K = DK + 2tKK. \quad (6)$$

The curvature of  $A_t$  is

$$F_t = dA_t + A_t A_t = tF^\sharp + (1-t)F + t(t-1)KK, \quad (7)$$

or

$$F_t = F + tDK + t^2KK = F + tD_tK - t^2KK. \quad (8)$$

Notice  $F_0 = F$ ,  $F_1 = F^\sharp$ . It follows that

$$\frac{dF_t}{dt} = DK + 2tKK = D_tK. \quad (9)$$

### III. THE COPY-STRUCTURE OF SPACE $\mathcal{A}$

The results of the previous section are valid for any two connections  $A$ ,  $A^\sharp$ . Let us address the question of copies. From (2), the necessary and sufficient condition to have  $F^\sharp = F$  is

$$DK + KK = 0. \quad (10)$$

From the Bianchi identity  $D^\sharp F^\sharp = 0$  applied with  $A^\sharp = A + K$  it follows that

$$[K, F] = 0. \quad (11)$$

These conditions [18] lead to the well-known determinantal conditions [19], [20] for the non-existence of copies. Notice that (3) and (11) imply  $DDK = 0$ . Copies are of interest only for non-abelian theories. In the abelian case  $KK \equiv 0$ ,  $DK \equiv dK$ , and condition (10) reduces to  $dK = 0$ , which means that locally  $K = d\phi$  for some  $\phi$ . Then  $A^\sharp = A + d\phi$ , a mere gauge transformation.

A first consequence of the conditions above is

$$\frac{dF_t}{dt} = D_tK = (1-2t)DK = (2t-1)KK.$$

A second consequence is that now the line through  $F$  and  $F^\sharp$  takes the form

$$F_t = F + t(t-1)KK = F + t(1-t)DK. \quad (12)$$

We have thus the curvatures of all the connections linking two copies along a line in connection space. Are there other copies on this line? In other words, is there any  $s \neq 0, 1$  for which  $F_s = F$ ? The existence of one such copy would imply, by the two expressions in

Eq.(12),  $DK = 0$  and  $KK = 0$ . But then, by the first equality of Eq.(8), all points on the line  $A_t = A + tK$  are copies. Three colinear copies imply that  $A_t$  is a line entirely formed of copies.

As  $DK = 0$  implies  $KK = 0$  by (10), it also implies  $D^\sharp K = 0$  [by (1)] and vice-versa. Consequently,

every point of the line  $A_t = tA^\sharp + (1 - t)A$  through two copies  $A$  and  $A^\sharp$  represents a copy when the difference tensor  $K = A^\sharp - A$  is parallel-transported by either  $A$  or  $A^\sharp$ .

In this case  $\frac{dF_t}{dt} = 0$ , that is,  $F_t = F$  for all values of  $t$ . Also,  $D_t K = 0$  for all  $t$ , so that  $K$  is parallel-transported by each connection on the line. Notice that an arbitrary finite  $K$  such that  $DK = 0$  does not necessarily engender a line of copies. It is necessary that  $K$  be *a priori* the difference between two copies.

The above condition is necessary and sufficient: if  $F_t \neq F$  for some  $t \neq 0, 1$ , Eqs.(12) imply both  $DK \neq 0$  and  $KK \neq 0$ . If the line joining two copies includes one point which is not a copy, then all other points for  $t \neq 0, 1$  correspond to non-copies.

*Given two copies and the straight line joining them, either there is no other copy on the line or every point of the line represents a copy.*

As a consequence, if there are copies for a certain  $F$ , and one of them (say,  $A$ ) is isolated, then there are no copies on the lines joining  $A$  to the other copies. Notice, however, that the existence of families of copies dependent on continuous parameters is known [21]. Thus, certainly not every copy is isolated.

The question of isolated copies is better understood by considering, instead of the above finite  $K$ , infinitesimal translations on  $\mathcal{A}$ . In effect, consider the variation of  $F$ ,  $\delta F = d\delta A + \delta A A + A \delta A = D_A \delta A$ . In order to have  $\delta F = 0$  it is enough that  $D_A \delta A = 0$ . Consequently, no copy is completely isolated. There can be copies close to any  $A$ : each variation satisfying  $D_A \delta A = 0$  leads to a copy. Taken together with what has been said above on the finite case, this means that there will be lines of copies along the “directions” of the parallel-transported  $\delta A$ ’s.

Notice that a line through copies of the vacuum is necessarily a line of copies. In effect, given  $A$  and  $A^\sharp$  with  $F = F^\sharp = 0$ , there is a gauge in which  $A = 0$  and another gauge in

which  $A^\sharp = 0$ . Using the first of these gauges,  $A_t = tK$  along the line. On the other hand,  $F_t = t(t-1)KK = 0$  by (12). As  $DK + KK = 0$ , we can write  $KK = DK + KK + KK = dK + AK + KA + KK + KK = dK + A^\sharp K + KA^\sharp = D^\sharp K = D^\sharp A^\sharp = F^\sharp = 0$ . It follows that  $F_t = 0$ .

Summing up, the overall picture is the following: from any  $A$  will emerge lines of three kinds:

- lines of copies, given by those  $\delta A$  which are parallel-transported by  $A$ ;
- lines of non-copies, given by those  $\delta A$  which are not parallel-transported by  $A$ ;
- lines along covariant matrix 1-forms  $K$  satisfying  $D_A K + KK = 0$ , which will meet one copy at  $A + K$ , and only that one.

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